Exam Modeling and Identification 2014-15

Datum	:	17-06-2015
Plaats	:	5161.0280
Tijd	:	14.00-17.00

Please motivate all your answers. The exam is OPEN BOOK.

1. Consider a system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad && x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y &= Cx, && y \in \mathbb{R}^p \end{aligned}$$

   which is controllable and observable. Assume that its transfer matrix $H(s) = C(Is - A)^{-1}B$ satisfies $H(s) = H(-s)$, $s \in \mathbb{C}$. (Such systems are sometimes called *time-reversible.*)

   (a) Prove that there exists an invertible $n \times n$ matrix P satisfying

   $$PA + AP = 0,\ PB = -B,\ C = CP$$

   such that furthermore $P^2 = I_n$ ($n \times n$ identity matrix).

   (b) Prove that $P^2 = I$ implies that there exists coordinates for $\mathbb{R}^n$ in which P takes the form

   $$P = \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}$$

   for a certain k with $0 \leq k \leq n$.

   (c) Prove that the result of part (b) implies that in such coordinates the system takes the form

   $$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

   (d) Confirm the above findings with the realization in controller form of $H(s) = \frac{1}{s^2+1}$.

2. Consider the transfer function $G(s) = \frac{1}{s+2}$.

   (a) Determine the Hankel singular value of $G(s)$. Take as approximation for $G(s)$ the zero transfer function $G_0(s) = 0$ corresponding to truncating the state space realization of $G(s)$ to the 'zero state system' $y = 0$. Compare the theoretical error bound between $G(s)$ and $G_0(s)$ based on truncation with the actual error bound.

(b) Compute a better approximation $y = du$ for $G(s)$ by explicitly minimizing the H_∞ norm

$$\sup_{\omega in \mathbb{R}} |G(i\omega) - d|$$

over all $d \in \mathbb{R}$. Also relate this outcome to the alternative to truncation studied in Homework 2, Exercise 2.

3. Consider the so-called ARMA (AutoRegressiveMovingAverage) model

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = a_{n+1} u(k-1) + \cdots + a_{n+m} u(k-m), \quad k \in \mathbb{Z}$$

with $u(j)$ and $y(j)$ the observed (measured) scalar inputs and outputs of the system at any time $j \in \mathbb{Z}$, while the coefficients a_i are unknown. If enough input and output data are available and the model is exact, then in principle it is easy to determine the coefficients by solving linear equations. Suppose however that the model is, as in most applications, *not* exact, in the sense that the coefficients are subject to random perturbations

$$a_i(k+1) = a_i(k) + w_i(k), \quad k \in \mathbb{Z}$$

with $w_i(k)$ discrete white noise processes with mean zero, and that the ARMA model is replaced by the so-called ARMAX model

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = a_{n+1} u(k-1) + \cdots + a_{n+m} u(k-m) + v(k), k \in \mathbb{Z}$$

with also $v(k)$ a discrete white noise process with mean zero, independent of $w(k)$.

We can write the identification problem of estimating the coefficients a_i as a Kalman filtering problem by defining the state vector $x(k)$ at time k, with components

$$x_i(k) := a_i(k), \quad i = 1, \cdots, n+m$$

satisfying the state dynamical equation

$$x(k+1) = x(k) + w(k)$$

where $w(k)$ is the vector with components $w_i(k), i = 1, \cdots, n+m$, and by rewriting the ARMAX model as

$$y(k) = H(k)x(k) + v(k)$$

with

$$H(k) := \begin{bmatrix} -y(k-1) & -y(k-2) & \cdots & -y(k-n) & u(k-1) & \cdots & u(k-m) \end{bmatrix}$$

Indicate the structure of the Kalman filter equations in this case.

4. Consider the scalar stochastic differential equation

$$\dot{x}_t = -2\alpha x_t + \sqrt{2\alpha} n_t$$

with n_t standard white noise, that is, a Gaussian process with zero mean and covariance function $R_n(t,s) = \delta(t-s)$.

(a) Show that the mean $\mu(t) = Ex_t$ satisfies the differential equation $\dot{\mu} = -2\mu$, and that the covariance $R_x(t) = E(x_t - \mu(t))^2$ satisfies

$$\dot{R}_x(t) = -2\alpha R_x(t) + 2\alpha$$

(b) Show that for initial conditions $\mu(0) = 0$ and $R_x(0) = 1$ the stochastic process x_t is stationary.

Distribution of points: Total 100; Free 10

1. a: 15, b: 5, c: 5, d: 5.

2. a: 10, b: 10.

3. 20.

4. a: 10, b: 10.